Pre-class Warm-up!!!

Which of the following functions has a frequency of 440 cycles per second?

a. sin 440t

b. sin 2π•440t ✓

c. sin (1/440t)

d. sin (1/(2π•440t))

e. sin 2π/440t

f. None of the above

→ Office hours ~~today~~ are cancelled

→ On Monday I will teach, then have office hours, on Zoom. The link is in an "Announcement" on the Canvas site, also on my home page, and also in an email you should all have got.

→ I will record Monday's class.

Section 5.6: Forced oscillations and resonance.

We have studied systems

- that oscillate (mass on the end of a spring):

$mx'' + kx = 0$ simple harmonic motion.

- with damping: under/over/critically damped

$mx'' + cx' + kx = 0$

With under damping $x(t) = e^{-\lambda t}\cos(\omega_0 t - \alpha)$

We now allow an external force to be applied:

$mx'' + cx' + kx = F(t)$

and we take F(t) to have the form

$$F(t) = A\cos\omega t + B\sin\omega t$$

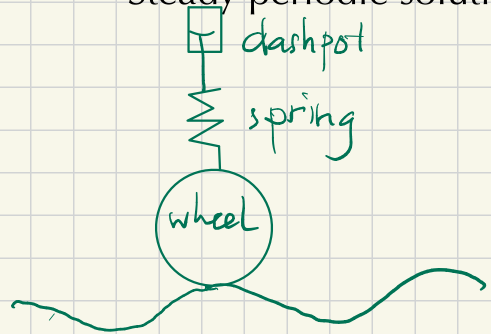
As already done: x(t) = x_c + x_p

x_c = "transient solution" = solution to homogeneous equation

x_p = "steady state solution"

New vocabulary:

- resonance, beats,
- Steady periodic solution, transient solution



We do not study:

- the particular models (cart with a flywheel)
- Static displacement, amplification factor
- Conservation of energy
- Practical resonance, critical resonance frequency (questions 15-18)
- adjusted forcing function (questions 7-10)

Section 5.6, question 2:
Express the solution of the initial value problem as a sum of two oscillations. Graph it.

$x'' + 4x = 5 \sin 3t, \quad x(0) = x'(0) = 0$

Solution: Char. poly $r^2+4 = (r+2i)(r-2i)$

$x_c = c_1 \cos 2t + c_2 \sin 2t$

Try $x_p = A \sin 3t + B \cos 3t$.

$x_p' = \ldots$

$x_p'' = \ldots$

Substitute, find A, B.

The solution is $x_c + x_p$

Apply the initial conditions to get c_1, c_2

Express $x_c = \cos(\omega_0 t - \alpha)$.

Section 5.6, question 2:
Express the solution of the initial value problem as a sum of two oscillations. Graph it.

$x'' + 4x = 5 \sin 3t, \qquad x(0) = x'(0) = 0$

Solution: Char. eqn. $r^2 + 4 = 0$, $r = \pm 2i$

$x_c = c_1 \cos 2t + c_2 \sin 2t$

To get x_p, try $x_p = A \cos 3t + B \sin 3t$

$x_p' = -3A \sin t + 3B \cos 3t$, $x_p'' = -9A \cos 3t - 9B \sin 3t$

$x_p'' + 4x_p = (4A - 9A) \cos 3t + (4B - 9B) \sin 3t$

$= 5 \sin 3t$ so $A = 0$, $-5B = 5$, $B = -1$

$x_p = -\sin 3t$

General solution $x = c_1 \cos 2t + c_2 \sin 2t - \sin 3t$

$x(0) = c_1 = 0$

$x' = -2c_1 \sin 2t + 2c_2 \cos 2t - 3 \cos 3t$

$x'(0) = 0 = 2c_2 - 3$, $c_2 = 3/2$

$x = \frac{3}{2} \sin 2t - \sin 3t = \frac{3}{2} \cos\left(2t - \frac{\pi}{2}\right) - \cos\left(3t - \frac{\pi}{2}\right)$

Question: what are the periods of these functions?

a. π/2, π/3

b. 1/2, 1/3

c. 2π, 3π

d. π, 2π/3 ✓

e. None of the above.

Resonance

Section 5.6, question 2 is:

$x'' + 4x = 5 \sin 3t, \quad x(0) = x'(0) = 0$

sin 2t, cos 2t are solutions to $x''+4x=0$

Modification:

$x'' + 4x = 4 \sin 2t, \quad x(0) = x'(0) = 0$

Question:

What is the best form of function to try so as to give a particular solution to $x'' + 4x = 4 \sin 2t$?

a. $y = A \sin 2t$

b. $y = A \sin 2t + B \cos 2t$

c. $y = A \sin 2t + B \cos 2t + Ct \sin 2t + Dt \cos 2t$

d. Something else.

I prefer $Ct \sin 2t + Dt \cos 2t$.

Proceed as before: find x_p

etc.

Resonance

Modification of question 2:

$x'' + 4x = 4\sin 2t, \qquad x(0) = x'(0) = 0$

Solution:

Again $x_c = a\cos 2t + b\sin 2t$

For a particular solution we try $x_p = At\cos 2t + Bt\sin 2t$

$x_p' = $ 4 terms

$x_p'' = $ 6 terms

$x_p'' + 4x_p = -4A\sin 2t + 4B\cos 2t = 4\sin 2t$

$A = -1,\ B = 0,\ x_p = -t\cos 2t$

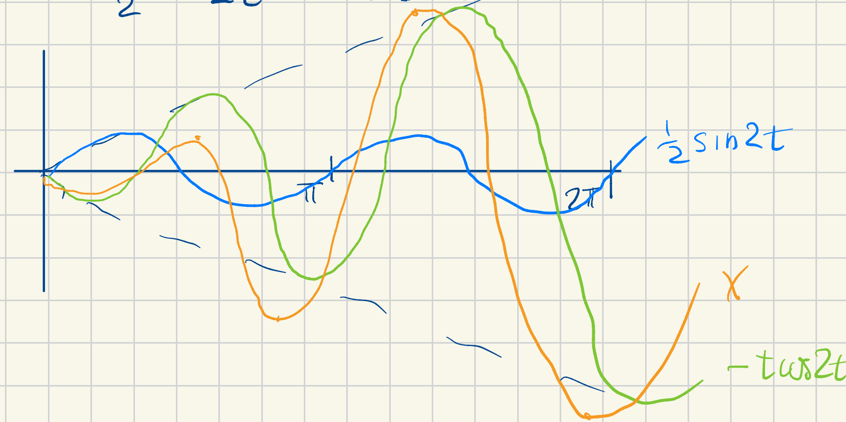
$x = c_1\cos 2t + c_2\sin 2t - t\cos 2t$

Initial conditions: $x(0) = c_1 = 0$

$x' = -2c_1\sin 2t + 2c_2\cos 2t - \cos 2t + 2t\sin 2t$

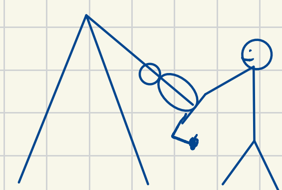
$x'(0) = 2c_2 - 1 = 0,\ c_2 = \frac{1}{2}$

$x = \frac{1}{2}\sin 2t - t\cos 2t$



The solution increases without bound

We have resonance. F(t) has the same frequency as a solution to the homog. eqn.



Beats occur when we have two solutions

$\cos(\omega_0 t)$, $\cos(\omega t)$ where

ω, ω_0 are close.

Write

$$x(t) = \cos(\omega_0 t) + \cos(\omega t)$$

$$= \frac{1}{2}\cos\left(\frac{\omega_0 t + \omega t}{2}\right)\cos\left(\frac{\omega_0 t - \omega t}{2}\right)$$

This comes from

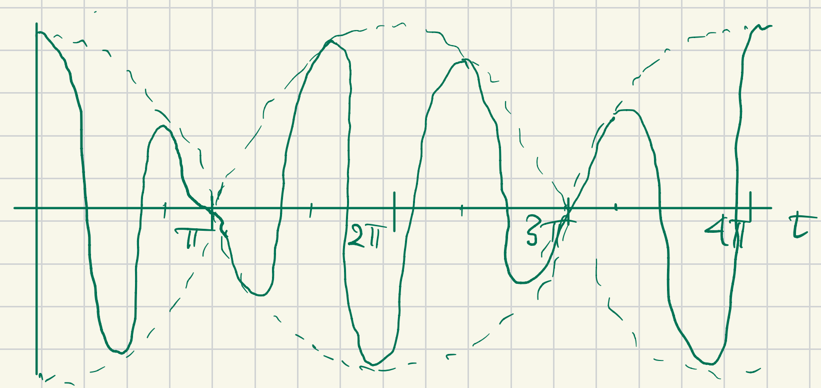
$$\cos(A+B) = \cos A\cos B - \sin A \sin B$$

$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$

$$\cos(A+B) + \cos(A-B) = 2\cos A\cos B$$

Example

$$\cos 2t + \cos 3t = \frac{1}{2}\cos\frac{5t}{2}\cos\frac{t}{2}$$



Take $A = \dfrac{\omega_0 t + \omega t}{2}$ $B = \dfrac{\omega_0 t - \omega t}{2}$

to get the previous formula.

5.6 question 12:
Find the steady periodic solution

$$x_{sp} = C\cos(\omega t - \alpha)$$

and the actual solution x(t) = x_sp(t) + x_tr(t) of
x'' + 6x' + 13x = 10 sin 5t, x(0) = x'(0) = 0

Solution $x_{tr} = x_c$ is a solution of

$$x'' + 6x' + 13x = 0$$

Char. eqn $r^2 + 6r + 13 = 0$, $r = \frac{-6 \pm \sqrt{36-52}}{2} = -3 \pm 2i$

$$x_{tr} = e^{-3t}(c_1 \cos 2t + c_2 \sin 2t)$$

For $x_p = x_{sp}$ try $x = A\sin 5t + B\cos 5t$

$x' =$

$x'' =$

$$A = \frac{-120}{1044} = \frac{-30}{261} \qquad B = \frac{-300}{1044} = \frac{-75}{261}$$

$$x_{sp} = \frac{-1}{261}(30\sin 5t + 75\cos 5t)$$

$$= \frac{-\sqrt{30^2 + 75^2}}{261}(\sin\alpha \sin 5t + \cos\alpha \cos 5t)$$

$$= -\frac{10}{6\sqrt{29}}\cos(5t - \alpha) \qquad \tan\alpha = \frac{2}{5}$$

So $x = e^{-3t}(c_1 \cos 2t + c_2 \sin 2t) - \frac{10}{6\sqrt{29}}\cos(5t - \alpha)$

$$\cos(-\alpha) = \frac{5}{\sqrt{29}}$$

$$x(0) = c_1 - \frac{5}{3\sqrt{29}} \cdot \frac{5}{\sqrt{29}} = 0, \quad c_1 = \frac{25}{3 \cdot 29}$$

$c_2 = \ldots$

$$x_{tr} = \frac{25}{6\sqrt{29}} e^{-3t} \cos(2t - \beta) \qquad \tan\beta = \frac{5}{2}.$$